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LETTER TO THE EDITOR

On an Ising model with multi-spin interactions

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Abstract. We prove that an Ising model with multi-spin interactions is equivalent to an Ising model with at most three-spin interactions on a decorated lattice and that an Ising model with only even-spin interactions is equivalent to an Ising model with only two-spin interactions on a decorated lattice.

An Ising model with multi-spin interactions has been discussed as a generalisation of the usual Ising model with pair interactions. Inequalities of spin correlation functions given by Griffiths (1967a, b, c) were studied for the general Ising model (Kelly and Sherman 1968, Ginibre 1969, Horiguchi and Morita 1979 and others). An Ising model with only one kind of multi-spin interaction has no phase transition and is always in the paramagnetic state (Mattis and Galler 1983, Horiguchi and Morita 1984). There is no spontaneous magnetisation at high temperature in an Ising model with only even-spin interactions (Horiguchi and Morita 1985). In the present letter, we prove that an Ising model with multi-spin interactions is in general equivalent to an Ising model with at most three-spin interactions on a decorated lattice.

We consider an Ising model with at most n-spin interactions ($n \geqslant 4$) on a finite set Λ of a d-dimensional lattice Z^d, whose Hamiltonian is given by

$$H = -\sum_{P \subset \Lambda} J_P \sigma_P. \tag{1}$$

Here we define σ_P for a subset P of Λ as follows

$$\sigma_P = \prod_{i \in P} \sigma_i. \tag{2}$$

σ_i is a usual Ising spin. J_P is an interaction constant for spins on sites belonging to the set P. We assume that J_P is zero unless $|P| \leqslant n$, where we denote by $|P|$ the cardinality of the set P. Furthermore we assume that the ranges of the interactions are finite: it is possible to find a number r such that J_P is zero whenever $\mathrm{diam}(P) > r$. We define the free energy of the system specified by the Hamiltonian H

$$f(H) = \lim_{|\Lambda| \to \infty} |\Lambda|^{-1} F(H), \tag{3}$$

$$F(H) = -kT \ln \sum_{\{\sigma\}} e^{-\beta H}, \tag{4}$$

where $\beta = 1/kT$ as usual and k is the Boltzmann constant. The thermodynamic limit is taken in the sense of van Hove (Ruelle 1969). We define a correlation function of

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spins σ_B for $B \subset \Lambda$.

$$\langle \sigma_B \rangle_H = \sum_{\{\sigma\}} \sigma_B \, e^{-\beta H} \left(\sum_{\{\sigma\}} e^{-\beta H} \right)^{-1} \tag{5}$$

We consider in the Hamiltonian H a term which is expressed by $-J_A\sigma_A$, where we assume $|A| \geq 2$. We rewrite the Hamiltonian H as follows

$$H = -\sum_{\substack{P \subset \Lambda \\ P \neq A}} J_P \sigma_P - J_A \sigma_A. \tag{6}$$

We separate the set A of sites into two subsets A_1 and A_2 of sites: $A = A_1 \cup A_2$, $A_1 \cap A_2 = \varnothing$, $A_1 \neq \varnothing$ and $A_2 \neq \varnothing$. We introduce a decorating site and denote it by d_A. We put an Ising spin σ_{d_A} on the site d_A. We introduce an Ising model which is described by the Hamiltonian defined as follows:

$$H' = -\sum_{\substack{P \subset \Lambda \\ P \neq A}} J_P \sigma_P - K_A \sigma_{d_A} (\mathrm{sgn}(J_A) \sigma_{A_1} + \sigma_{A_2}), \tag{7}$$

where

$$K_A = \frac{1}{2\beta} |\cosh^{-1} e^{2\beta |J_A|}|. \tag{8}$$

βK_A tends to $(\beta |J_A|)^{1/2}$ as β tends to zero. Then we have

$$F(H) = kT \ln 2 + |J_A| + F(H') \tag{9}$$

and

$$\langle \sigma_B \rangle_H = \langle \sigma_B \rangle_{H'}. \tag{10}$$

We call this transformation an inverse decimation transformation (or a decoration transformation). By this transformation, we are able to reduce the number of spins which interact with each other if $|A| \geq 4$. However if $|A| = 3$, the three-spin interaction is expressed by a two-spin interaction and a three-spin interaction. If $|A| = 2$, the two-spin interaction is expressed by the two two-spin interactions. We note here that this transformation conserves the total number of antiferromagnetic interactions in the Hamiltonians obtained by its successive applications.

Next we introduce another type of inverse decimation transformation. We assume $|A| \geq 4$ in (6). We separate the set A of sites into four subsets A_1, A_2, A_3 and A_4 of sites: $A = A_1 \cup A_2 \cup A_3 \cup A_4$, $A_i \cap A_j = \varnothing$ for $i \neq j$ and $A_i \neq \varnothing$ where i and j belong to $\{1, 2, 3, 4\}$. We introduce an Ising model which is described by the following Hamiltonian:

$$\begin{aligned} H' = &-\sum_{\substack{P \subset \Lambda \\ P \neq A}} J_P \sigma_P - I_A \sigma_{d_A} (\sigma_{A_1} + \sigma_{A_2} + \sigma_{A_3} - \mathrm{sgn}(J_A) \sigma_{A_4}) \\ &- \mathrm{sgn}(J_A) I'_A \sigma_{A_4} (\sigma_{A_1} + \sigma_{A_2} + \sigma_{A_3}) + I'_A (\sigma_{A_1}\sigma_{A_2} + \sigma_{A_2}\sigma_{A_3} + \sigma_{A_3}\sigma_{A_1}), \end{aligned} \tag{11}$$

where I_A and I'_A are positive and given as follows

$$I_A = 2|J_A| + (1/2\beta) \ln[(1 + f_A)^{1/2} (1 + f_A^{1/2})], \tag{12}$$

$$I'_A = |J_A| + (1/4\beta) \ln(1 + f_A), \tag{13}$$

$$f_A = (1 - e^{-8\beta |J_A|})^{1/2}. \tag{14}$$

βI_A and $\beta I'_A$ tend to $(\beta |J_A|/2)^{1/4}$ and $(\beta |J_A|/2)^{1/2}$, respectively, as β tends to zero.

Then we have

$$F(H)=kT\ln D_A+F(H') \tag{15}$$

and

$$\langle\sigma_B\rangle_H=\langle\sigma_B\rangle_{H'} \tag{16}$$

where

$$D_A=2\,\mathrm{e}^{3\beta|J_A|}(1+f_A)^{1/2}. \tag{17}$$

In this transformation, the total number of antiferromagnetic interactions is not covered any more by its successive applications.

As seen above, by successive applications of transformation given by (7) or (11), we are able to reduce the maximum number of spins which interact with each other to two or three. There are many ways of reduction. As an example, we give a way of reduction obtained by requiring the total number of decorated sites minimum and the conservation of the number of antiferromagnetic interactions. By using (7), a multi-spin interaction is reduced to three-spin interactions. More precisely an n-spin interaction is reduced to $n-2$ three-spin interactions by $n-3$ applications. A case of six-spin interactions is shown in figure 1. We write the Hamiltonian H as follows:

$$H=H_1+H_2+H_3+\Delta H \tag{18}$$

where

$$\Delta H=-\sum_{\substack{P\subset\Lambda\\|P|\geqslant4}}J_P\sigma_P. \tag{19}$$

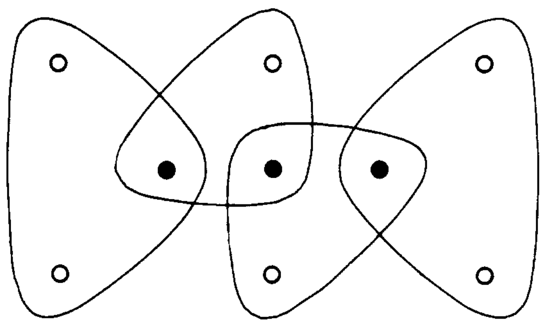

Figure 1. We assume that there exists a six-spin interaction between spins on six lattice sites shown by the open circles. The closed circles express the decorating lattice sites. The spins on three lattice sites enclosed by a full line interact with each other by an effective three-spin interaction. The six-spin interaction is reduced to four three-spin interactions.

Here H_1, H_2 and H_3 are the terms of external field, two-spin interactions and three-spin interactions, respectively. Then we have

$$F(H)=\sum_{\substack{P\subset\Lambda\\|P|\geqslant4}}\sum_{l=1}^{l_P}(kT\ln2+K_P^{(l-1)})+F(\tilde{H}) \tag{20}$$

and

$$\langle\sigma_B\rangle_H=\langle\sigma_B\rangle_{\tilde{H}}. \tag{21}$$

Here $l_P = |P| - 3$, and $\tilde{H}$ is defined as follows

$$\tilde{H} = H_1 + H_2 + H_3 + \tilde{H}_3 \tag{22}$$

$$\tilde{H}_3 = -\sum_{\substack{P\subset\Lambda \\ |P|\geq 4}} \left(\operatorname{sgn}(J_P) K_P^{(1)} \sigma_{d_P^{(1)}} \sigma_{p^{(1)}} \sigma_{p^{(2)}} + K_P^{(l_P)} \sigma_{d_P^{(|P|)}} \sigma_{p^{(|P|-1)}} \sigma_{p^{(|P|)}} \right.$$
$$\left. + \sum_{l=1}^{l_P-1} K_P^{(l+1)} \sigma_{d_P^{(l)}} \sigma_{d_P^{(l+1)}} \sigma_{p^{(l+2)}} \right), \tag{23}$$

where we denote the set P by $\{p^{(1)}, p^{(2)}, \ldots, p^{(|P|)}\}$ and the decorating sites by $d_P^{(l)}$ with $l = 1, 2, \ldots, l_P$. $K_P^{(l)}$ are given as follows:

$$K_P^{(l)} = (1/2\beta)|\cosh^{-1} \mathrm{e}^{2\beta K_P^{(l-1)}}|, \tag{24}$$

$$K_P^{(0)} = |J_P|. \tag{25}$$

In the limit as β tends to zero, we have

$$\beta K_P^{(l)} = (\beta|J_P|)^{1/2^l}(1 + \mathrm{O}(\beta|J_P|)). \tag{26}$$

The total number of the decorating sites is $\Sigma_{P\subset\Lambda}(|P|-3)$ which is $\mathrm{O}(|\Lambda|)$. Then the thermodynamic limit of the free energy exists in (3) and (20). In this way, the Ising model with multi-spin interactions is equivalent to an Ising model with at most three-spin interactions on a decorated lattice.

We give some examples. The first one is an Ising model with two-spin interactions and four-spin interactions on the square lattice. The Hamiltonian is given by

$$H = H_2 - J_4 \sum_{(i,j)} \sigma_{i,j}\sigma_{i+1,j}\sigma_{i,j+1}\sigma_{i+1,j+1}, \tag{27}$$

where H_2 is the term of two-spin interactions. (i, j) denotes a site on the square lattice. When H_2 consists of only next-nearest-neighbour interactions, the system was solved by the eight-vertex model (Baxter 1971, Wu 1971, Kadanoff and Wegner 1971). When H_2 consists of only one kind of two-spin interactions, some of the systems were investigated in a previous paper (Horiguchi 1985). From (7), we have

$$\tilde{H} = H_2 - K_4 \sum_{(i,j)} \sigma_{i+1/2,j+1/2}(\operatorname{sgn}(J_4)\sigma_{i,j}\sigma_{i,j+1} + \sigma_{i+1,j}\sigma_{i+1,j+1}), \tag{28}$$

where K_4 is obtained from (8) by substituting J_4 into J_A. $(i+\frac{1}{2}, j+\frac{1}{2})$ denotes a decorating lattice site. If we apply (11), we have

$$\tilde{H} = H_2 - \sum_{(i,j)} \{ I_4 \sigma_{i+1/2,j+1/2}(\sigma_{i,j} + \sigma_{i+1,j} + \sigma_{i,j+1} - \operatorname{sgn}(J_4)\sigma_{i+1,j+1})$$
$$+ I_4' \operatorname{sgn}(J_4)\sigma_{i+1,j+1}(\sigma_{i,j} + \sigma_{i+1,j} + \sigma_{i,j+1})$$
$$- I_4'(\sigma_{i,j}\sigma_{i+1,j} + \sigma_{i+1,j}\sigma_{i,j+1} + \sigma_{i,j+1}\sigma_{i,j})\} \tag{29}$$

where I_4 and I_4' are obtained from (12), (13) and (14). This is Jüngling's formulation of the eight-vertex model (Jüngling and Obermair 1974, Jüngling 1975). Both systems described by (28) and (29) are equivalent to the system described by (27). The second example is an Ising model with star interactions on the square lattice as shown in figure 2(a):

$$H = H_2 - J_5 \sum_{(i,j)} \sigma_{i,j}\sigma_{i+1,j}\sigma_{i-1,j}\sigma_{i,j+1}\sigma_{i,j-1}. \tag{30}$$

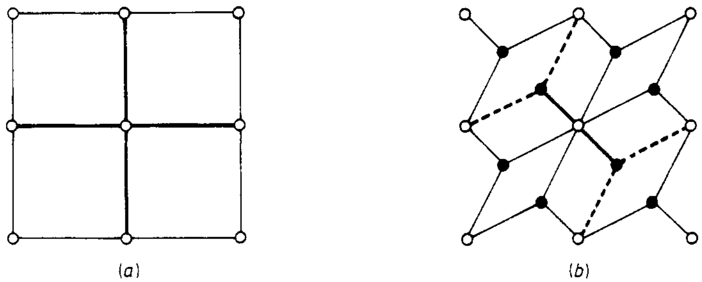

Figure 2. (a) A part of the square lattice is shown. The lattice sites are shown by the open circles. A star interaction is assumed between five spins on the lattice sites connected by bold full lines. (b) The star interaction shown in (a) is reduced to three three-spin interactions which are shown by the bold broken lines or the bold full lines. The full circles express the decorating lattice sites. When each open circle is connected with its nearest and next-nearest six full circles as shown here, then all the lattice sites make a diced lattice.

This system is equivalent to an Ising model with three-spin interactions on a diced lattice shown in figure 2(b):

$$\tilde{H} = H_2 - \sum_{(i,j)} (\operatorname{sgn}(J_5) K_5^{(1)} \sigma_{i-1/3,j+1/3}\sigma_{i-1,j}\sigma_{i,j+1} + K_5^{(2)}\sigma_{i-1/3,j+1/3}\sigma_{i,j}\sigma_{i+1/3,j-1/3} + K_5^{(2)}\sigma_{i+1/3,j-1/3}\sigma_{i+1,j}\sigma_{i,j-1}), \tag{31}$$

where $K_5^{(1)}$ and $K_5^{(2)}$ are obtained from (24) and (25). $(i-\frac{1}{3}, j+\frac{1}{3})$ and $(i+\frac{1}{3}, j-\frac{1}{3})$ denote decorating lattice sites. When $H_2 = 0$, this system shows no phase transition and is always in the paramagnetic state (Mattis and Galler 1983).

Finally we make three remarks. The first one is that an Ising model with even-spin interactions in an external field is equivalent to an Ising model with only two-spin interactions on a decorated lattice and with the external field applied on only sites of original lattice; that is confirmed by successive applications of (11). Then by using theorem 1 proved in a previous paper (Horiguchi and Morita 1979) and the results obtained by Griffiths (1967c), we have that an Ising model with even-spin interactions has no spontaneous magnetisation at high temperatures when the ranges of interactions are finite (see also Morita and Horiguchi 1985, Horiguchi and Morita 1985). The second is the one for an Ising model of arbitrary spin with multi-spin interactions. Griffiths (1969) proved that an Ising spin of spin $\frac{1}{2}p$, which is greater than $\frac{1}{2}$, is expressed in terms of a cluster of p Ising spins of $\frac{1}{2}$ interacting among themselves through suitable ferromagnetic two-spin interactions. By using his result, the Ising model of arbitrary spin with multi-spin interactions is reduced to an Ising model of spin ± 1 with multi-spin interactions, and then to an Ising model of spin ± 1 with at most three-spin interactions on a decorated lattice. The last remark is that the arguments given in this letter are also applied to Ising models on a set of sites which does not construct a regular lattice.

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